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Geometrical representation of angular momentum coherence and squeezing

Abir Bandyopadhyay* and Jagdish Rai†
Department of Physics, Indian Institute of Technology, Kanpur - 208016, INDIA.
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Abstract

A simple, and elegant geometrical representation is developed to describe the concept of coherence and squeezing for angular momentum operators. Angular momentum squeezed states were obtained by applying Bogoliubov transformation on the angular momentum coherent states in Schwinger representation [Phys. Rev. **A 51** (1995)]. We present the geometrical phase space description of angular momentum coherent and squeezed states and relate with the harmonic oscillator. The unique feature of our geometric representation is the portraying of the expectation values of the angular momentum components accompanied by their uncertainties. The bosonic representation of the angular momentum coherent and squeezed states is compared with the conventional one mentioning the advantages of this representation of angular momentum in context of coherence and squeezing. Extension of our work on single mode squeezing to double mode squeezing is presented and compared with the single mode one. We also point out the possible applications of the geometrical representation in analyzing the accuracy of interferometers and in studying the behavior, dynamics of an ensemble of quantum-mechanical two-level systems and its interaction with radiation.

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*Present address : Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560 012, INDIA. Also at Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore 560 064 (Do **NOT** communicate to this JNC ASR address.) Electronic address :abir@cts.iisc.ernet.in

†Electronic addresses :jrai@iitk.ac.in, jrai@rediffmail.com

I. INTRODUCTION

The harmonic oscillator phase space description of electromagnetic fields has had great success in understanding the semiclassical and quantum theories of coherent light [1]. Coherent states are defined generally as minimum uncertainty states. The product of the uncertainties in the quadratures is minimum for these states and both the quadratures have equal uncertainties. Squeezing redistributes the uncertainties of the two quadratures resulting in one of them having less than its value for the coherent state at the expense of increasing the other [2]. The geometrical description of the Heisenberg uncertainty relation of two noncommuting variables (quadratures) is well known to give a better understanding of the inherent fluctuation due to the quantum nature of the squeezed light. The quadratures of the phase space retain the minimum uncertainty property of the coherent states.

Angular momentum or SU(2) algebra describes the behavior of an ensemble of two quantum-level noninteracting systems. Examples of these systems include interferometers, non-interacting ensemble of two-level atoms or molecules. Sensitivity of interferometers, when described by SU(2) algebra, can be defined in terms of matrix elements of mean and variances of angular momentum operators [3]. Feynman *et al.* [4] have constructed a simple geometrical representation of the Schrodinger equation for such two-level systems to solve maser problems and radiation damping. In their approach the two-level quantum system is described by the state vector $\vec{r}$ with components determined by the probability amplitude. The dynamics of the system is described by the differential equation of the state vector $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$, where $\vec{\omega}$ is the angular velocity. However, in their approach there is no description of the uncertainty associated with the state vector. These uncertainties are important in describing the full quantum nature of the system e.g., the interaction of the system with quantized radiation. Our work is an extension of the Feynman *et al.*'s description as we have included the uncertainties associated with the state vectors. Considerable amount of work has also been done in order to understand the coherent angular momentum states [6,7]. Arecchi had constructed the angular momentum coherent states by rotating the angular momentum ground state [7] similarly as of displacement operator coherent states for electromagnetic waves. During last few years several authors have tried to construct squeezed angular momentum states and study their properties in terms of atomic systems [8–10].

Schwinger developed an abstract algebra which perfectly describes the angular momentum systems [5]. The algebra assumes the angular momentum operators as a combination of two sets ($\pm$) of uncorrelated creation and annihilation operators of spin $\pm\frac{1}{2}$ which obey the bosonic commutation relations. Apart from the usual rotational angular momentum states, this algebra also describes the pseudo-angular momentum systems such as interferometry or a collection of two level systems proving to be quite a success. Atkins and Dobson (AD) had constructed angular momentum coherent (SAMC) states using Schwinger representation about one and a half year before Arecchi [11]. They had connected these states to two dimensional oscillator states quantized in orthogonal direction. The matrix elements of the components can be calculated easily from the expressions of their operator form. In the classical limit the SAMC states become the classical vector $\vec{J}$ in the sense that they behave like their classical analogs and their variances are small compared to the absolute value of their averages. They have used these states to study elliptically polarized states successfully

[12]. These SAMC states have also been used to describe the rotational levels of nuclear and moleculer systems by Fonda itet al. [13].

Recently, we have proposed a technique for the generation of angular momentum squeezed states [14]. By combining the Schwinger angular momentum representation with boson operators and the concept of squeezing of bosons via Bogoliubov transformation, we were successful in producing states that exhibit squeezing of angular momentum operators. We have called these states Schwinger Angular Momentum Squeezed (SAMS) states. Our procedure was an extension of the work of Atkins and Dobson [11] on angular momentum coherent (SAMC) states. We also found application of the SAMS states in enhancing the sensitivity of interferometers and in study of two level atoms [14]. In this paper we have constructed an elegant geometrical representation of the concept of coherence and squeezing for the angular momentum operators. We present some new results for two mode squeezing and discuss their geometrical behavior and non-usefulness in this paper. The results of SAMC states and single mode SAMS states are also considered for the geometrical phase space realization of these states. Our representation, combined with the picture given by Feynman *et al.*, ensures to be an important tool to study the interaction between radiation and coherent or squeezed two-level systems.

The paper is organized as follows. In sec. II we recapitulate some results of the Schwinger Angular Momentum Coherent (SAMC) states to describe them geometrically and compare them with the bosonic counterpart. In the next section (III) we present the new two mode squeezing results (IIIb) along with relevant single mode SAMS state results (IIIa) to understand the phase space geometry. Finally we conclude comparing our technique with other works on coherent angular momentum in context of geometrical understanding of the phase space. We also furnish the possible applications of the geometrical representation in the conclusion.

II. ANGULAR MOMENTUM COHERENT STATES

Schwinger [5] developed the entire angular momentum algebra in terms of two sets(up and down) of uncorrelated harmonic oscillator creation and annihilation operators constructing the angular momentum operators as

$$J_+ = J_x + iJ_y = {a_+}^\dagger a_- \tag{2.1a}$$

$$J_- = J_x - iJ_y = {a_-}^\dagger a_+ \tag{2.1b}$$

$$J_z = \frac{1}{2}({a_+}^\dagger a_+ - {a_-}^\dagger a_-). \tag{2.1c}$$

The operators ${a_\pm}^\dagger(a_\pm)$ create (annihilate) a $\pm\frac{1}{2}$ spin and follow the bosonic commutation relations which means that the whole system is considered as a combination of two sets of boson states. This construction satisfies the standard angular momentum commutation relation $[J_l, J_m] = i\epsilon_{lmn}J_n$. The angular momentum basis states $|j, m\rangle$ can thus be created by action of the oscillator operators on the vacuum spinor $|0, 0\rangle$

$$|j,m\rangle = |j+m\rangle_+ \otimes |j-m\rangle_- = [(j+m)!(j-m)!]^{-\frac{1}{2}}(a_+{}^\dagger)^{j+m}(a_-{}^\dagger)^{j-m}|0,0\rangle. \qquad (2.2)$$

AD [11] constructed the Schwinger Angular Momentum Coherent (SAMC) states as the simultaneous eigenstates of the operators $a_\pm$. They have shown that the SAMC states are minimum uncertainty states for the angular momentum-angle uncertainty relation in the large $N(N = n_+ + n_- \geq 10)$ limit [11]. According to their definition the angular momentum coherent states $|\tilde{\alpha}\rangle = \tilde{D}|0,0\rangle$, with $\tilde{D} = D_+(\alpha_+)D_-(\alpha_-)$, obey,

$$a_\pm|\tilde{\alpha}\rangle = a_\pm|\alpha_+,\alpha_-\rangle = \alpha_\pm|\alpha_+,\alpha_-\rangle \qquad (2.3)$$

The expansion of $|\tilde{\alpha}\rangle$ in terms of angular momentum basis is given by

$$|\tilde{\alpha}\rangle = e^{-\frac{1}{2}N}\sum_j\sum_m(2j!)^{-\frac{1}{2}}\begin{pmatrix} 2j \\ j+m \end{pmatrix}^{\frac{1}{2}}\alpha_+{}^{j+m}\alpha_-{}^{j-m}|j,m\rangle \qquad (2.4)$$

The ∞:1 mapping of $\tilde{\alpha}$ onto $(\langle J_x\rangle,\langle J_y\rangle,\langle J_z\rangle)$ is a consequence of the 2:1 homomorphism of SU(2) and SO(3) [15]. SU(2) is spanned by the subset of spinors of length $\sqrt{N}$ and SO(3) is spanned by the subset of vectors of length $\langle J\rangle$. The spinor $|\tilde{\alpha}\rangle$ is a vector sum of different vectors $|j,m\rangle$ in the physical angular momentum space. Once the four parameters in $\alpha_\pm$ are fixed, it automatically fixes the values of j and m in the angular momentum space. To calculate the mean and variances of the angular momentum components one has to use the expressions of them in SAMC basis and fix the corresponding parameters. The matrix elements can be expressed in terms of any set of these parameters ($\alpha_\pm$ or j and m). The expressions in terms of the angular momentum parameters give better understanding in the physical space. For this reason we have expressed the matrix elements in terms of angular momentum parameters throughout the paper.

The mean of the angular momentum components are calculated as following

$$\langle J_x\rangle = \sqrt{j^2 - m^2}\cos\Theta \qquad (2.5a)$$

$$\langle J_y\rangle = \sqrt{j^2 - m^2}\sin\Theta \qquad (2.5b)$$

$$l\angle J_z\rangle = m \qquad (2.5c)$$

where $\Theta = (\theta_+ - \theta_-)$ and the variances are

$$\Delta J_x{}^2 = \Delta J_y{}^2 = \Delta J_y{}^2 = \frac{1}{2}j \qquad (2.6)$$

The above results show that the average value of $\vec{J}$ *i.e.* the tip of it lies on a sphere of radius j. This can easily be verified by squaring and adding the mean values of the angular momentum components. The fluctuations of the components are also same in the three directions. The equation of the region of uncertainty creates a sphere of radius $\sqrt{\frac{3j}{2}}$ about the tip of the vector. This is shown in Fig. 1. The polar angle of the state vector in the three dimensional phase space is actually realized to be Θ, the difference between two phase angles of the coherence parameters.

We have shown the uncertainties of harmonic oscillator and a schematic of In three dimensional phase space we have shown the uncertainty sphere of the angular momentum vector in Fig. 1(c). This is also compared with the uncertainty circle of the harmonic oscillator in two dimensional phase space in Fig. 1(a). For SAMC states the radii of these uncertainty spheres ($=\sqrt{\frac{3j}{2}}$) depend only on the radii of the mean sphere ($=j$) on which the tip of the vector lies. For a fixed j value the radii of the uncertainty spheres does not vary on its position on the sphere for the choice of the parameters m or Θ. The uncertainty spheres has a circular projection (uncertainty circle) in the X-Y plane. We have shown some uncertainty spheres on it for some different positions (different values of m and Θ) in Fig. 1(c). The positions for $|m|\langle j$ can be compared with the displacement of the ground state in the phase space picture for harmonic oscillator. The displacement of the uncertainty circle in the phase space of harmonic oscillator is performed by the rotation of the uncertainty sphere in corresponding three dimensional phase space for SAMC states. The position after rotation is governed by the values of the parameters m and $(\theta_+ - \theta_-)$.

The uncertainty relation corresponding to the commutation relation between the components of the angular momentum, $[J_l, J_m] = i\epsilon_{lmn}J_n$, is

$$\Delta {J_x}^2 \Delta {J_y}^2 \geq \frac{1}{4}|\langle J_z\rangle|^2 \tag{2.7}$$

Putting the expressions of the matrix elements in the last equation one can check that the equality occurs at $m=\pm j$. The two solutions for the equality show the SU(2) symmetry of the system. Other values of the angular momentum projection have uncertainties of both the quadratures equal and same as the extremum cases. Though the non-extremum cases does not violate the uncertainty relation in the last equation but the relation is an equality only for extremum cases. Anyway, for physical purposes we are interested in the absolute uncertainties in the quadratures which remain same for all the SAMC states with same j value.

III. ANGULAR MOMENTUM SQUEEZED STATES

Following the work of AD on angular momentum coherent states, we have generated squeezed angular momentum states by operating the squeezing operators of the bosonic states on the SAMC states [14]. For the two mode ($\pm$) bosonic case the squeezing operators can be defined as

$$S_\pm(\xi_\pm) = exp\left[\frac{1}{2}(\xi_\pm {a_\pm^\dagger}^2 - {\xi_\pm}^* {a_\pm}^2)\right], \quad \xi_\pm = r_\pm e^{i\phi_\pm} \tag{3.1}$$

We have created the general SAMS states by operating the two mode squeezing operator $\tilde{S} = S_+(\xi_+)S_-(\xi_-)$ on the SAMC states

$$|\psi\rangle = S_+(\xi_+)S_-(\xi_-)D_+(\gamma_+)D_-(\gamma_-)|0,0\rangle \tag{3.2}$$

For convenience in calculation we use the relation to interchange the order of the squeezing and displacement operators [16] and write the general SAMS states as

$$|\psi\rangle = D_+(\alpha_+)D_-(\alpha_-)S_+(\xi_+)S_-(\xi_-)|0,0\rangle \tag{3.3}$$

where $\gamma_\pm = \cosh r_\pm \alpha_\pm + e^{i\phi_\pm}\sinh r_\pm {\alpha^*}_\pm$.

A. Single Mode Squeezing

The calculation of the expectation values and variances of the operators of our interest is cumbersome due to the dependence on the large number (eight) of parameters involved in the general SAMS states. First we consider the squeezing in one mode only. At this point we do this for simplicity though the utility of this choice will be clear in the following subsection. The SU(2) symmetry tells us that we can choose any one of the modes for squeezing. So we choose to squeeze in the + mode which reduces the expression of the basis state vectors of single mode SAMS states to

$$|\Psi\rangle = \tilde{D}S(\xi)|0,0\rangle \tag{3.4}$$

where we have dropped the unnecessary suffix +.

Calculating the mean of the angular momentum components [14]

$$\langle J_x\rangle = \sqrt{j^2 - m^2}\cos\Theta \tag{3.5a}$$

$$\langle J_y\rangle = \sqrt{j^2 - m^2}\sin\Theta \tag{3.5b}$$

$$\langle J_z\rangle = m + \frac{1}{2}\sinh^2 r \tag{3.5c}$$

we see that the traversing spherical surface of the mean of the tip of the angular momentum vector has been changed to a prolate ellipsoid with same axes in X and Y. The expression for the mean value of the angular momentum projection or the Z-axis of the ellipsoid show an increase as squeezing is increased. This is shown in Fig. 1(d). It is to be noted that squeezing in the other mode will change the mean sphere to an oblate ellipsoid with $\langle J_z\rangle = m - \frac{1}{2}\sinh^2 r_-$ instead of a prolate one. However, the other two axes will not change due to the choice of mode of squeezing. We calculated the variances to get a feel of the uncertainty nature of the SAMS states as

$$\Delta {J_x}^2 = \frac{1}{2}j + \frac{1}{2}\sinh r[\frac{1}{2}\sinh r\{1 + 2(j-m)\} + (j-m)\cosh r\cos\delta] \tag{3.6a}$$

$$\Delta {J_y}^2 = \frac{1}{2}j + \frac{1}{2}\sinh r[\frac{1}{2}\sinh r\{1 + 2(j-m)\} - (j-m)\cosh r\cos\delta] \tag{3.6b}$$

$$\Delta {J_z}^2 = \frac{j+m}{4}[e^{2r}\cos^2\eta + e^{-2r}\sin^2\eta] + \frac{1}{2}\sinh^2 r\cosh^2 r + \frac{j-m}{4} \tag{3.6c}$$

where $\delta = 2\theta_- - \phi$ and $\eta = \theta_+ - \frac{\phi}{2}$. r and ϕ are the squeezing parameters of the + mode. The effect of squeezing in the other mode on the variances of the angular momentum components can be obtained from the last equation by interchanging the suffixes which is a consequence of the SU(2) symmetry. The uncertainty ellipsoid for $m = -j$ have been shown in Fig. 1(d). It is to be noted that all the axes of this uncertainty ellipsoid are different. This results to an ellipsoidal projection on the X-Y plane. The squeezing of angular momentum can be compared with the harmonic oscillator squeezing, which is shown in Fig.1(b).

In Fig. 2 we have plotted the dependence of the axes of the projected uncertainty ellipses on the squeezing parameter r. The projected uncertainty circle on the X-Y plane for the SAMC states are transformed to ellipses, but with greater area (uncertainty product). It is clear from the expressions that the maximum squeezing *i.e.* minimum fluctuation of the squeezed quadrature occurs to the minimum uncertainty circle at $m = \pm j$ as expected physically. From Fig. 2 it is clear that the minimum uncertainty circle is squeezed (length of the semiminor axis is reduced) up to a critical value of r ($=r_{min}$) though its area (uncertainty product) is increased throughout. After that critical value of r the length of both the axes of the uncertainty ellipse increases.

It is interesting to note that squeezing in the + or - mode results squeezing of uncertainty in J_y and J_x respectively. This means that the squeezing in the angular momentum quadratures are directly related to the mode of squeezing. It will be interesting to express the squeezing operators in terms of operators in X-Y coordinates instead of $\pm$ to identify the reason and exact mapping between them.

B. Double Mode Squeezing

Now we consider the case of double mode squeezing. In the last subsection we have squeezed only in one mode for the sake of simplicity and promised to give the practical reasoning for this simplification in this subsection. Actually two mode squeezing does not help in reducing the uncertainty of any of the quadratures which we will show now. We can claim from the results of the last subsection that if we squeeze both the modes the uncertainties of the quadratures will be squeezed and expanded simultaneously. The squeezing of the second mode in effect reduce the amount of squeezing achieved by the first mode squeezing.

We have calculated the expectation values of the components of the angular momentum for double mode squeezing as

$$\langle J_x \rangle = \sqrt{j^2 - m^2} \cos\Theta \tag{3.7a}$$

$$\langle J_y \rangle = \sqrt{j^2 - m^2} \sin\Theta \tag{3.7b}$$

$$\langle J_z \rangle = m + \frac{1}{2}(\sinh^2 r_+ - \sinh^2 r_-). \tag{3.7c}$$

The expectations of the angular momentum components in X and Y direction are seen to be same as that of SAMC states with no effect of squeezing. The mean sphere is clearly seen to be transformed to an ellipsoid in general with same X and Y axes. The Z axis of the ellipsoid will increase or decrease as difference of squares of the hyperbolic sine functions of the two parameters $r_\pm$. This affects the shape of the mean spheroid to prolate or oblate. However, the expectation value of J_z can be made to be same as that of SAMC states by squeezing both the modes equally. This will make the mean ellipsoid to be same mean sphere as for SAMC states.

To show that the effect of double mode squeezing does not help in squeezing of the angular momentum quadratures we have calculated the uncertainties for some special choice of parameters. We have chosen the phases in the squeezing parameters to be equal to zero and the magnitudes of the squeezing parameters to be equal to r. This choice does not affect the basic motivation of representing the states geometrically or prove the disadvantage of double mode squeezing. We have calculated the uncertainties in the angular momentum components for this special [18] choice as

$$\Delta {J_x}^2 = \frac{1}{2}[j + \sinh r\{\sinh r + \cosh r \cos(2\theta_+)\}(j+m) \\ + \sinh r\{\sinh r + \cosh r \ \cos(2\theta_-)\}(j-m) + \sinh^2 r(1+\cosh 2r)] \tag{3.8a}$$

$$\Delta {J_y}^2 = \frac{1}{2}[j + \sinh r\{\sinh r - \cosh \ r \ \cos(2\theta_+)\}(j+m) \\ + \sinh r\{\sinh r - \cosh r \cos(2\theta_-)\}(j-m) + \sinh^2 r(1+\cosh 2r)] \tag{3.8b}$$

$$\Delta {J_z}^2 = \frac{j+m}{4}[e^{2r}\cos^2\theta_+ + e^{-2r}\sin^2\theta_+] + \frac{j-m}{4}[e^{2r}\cos^2\theta_- + e^{-2r}\sin^2\theta_-] \\ +\sinh^2 r \ \ \cosh^2 r \tag{3.8c}$$

The uncertainty ellipsoids on the mean ellipsoid are similar as single mode squeezing case. The uncertainties of the quadratures and their product are plotted in Fig. 3 for j=50, m=-50, and $\theta_+ = \theta_- = 0$. Here we have plotted the results for the extremum projection states which drops out the first term in both the quadrature uncertainties. The squeezing in the uncertainty of J_y prove our claim that double mode squeezing deteriorates the effect of single mode squeezing which was expected from qualitative reasoning. Squeezing both the modes by same amount retains the SU(2) symmetry of the system but the choice of the phases and magnitudes of coherent parameters breaks it resulting different expressions and curves for the quadratures. This choice has been made to show the difference distinctly and the dependence on the coherence phases. With all the parameters same for the two modes one can show that the two quadratures will behave similarly. We do not present the geometrical pictures for the double mode squeezing as they are similar to the single mode squeezing.

IV. CONCLUSION

We have represented the angular momentum coherent (SAMC) and squeezed (SAMS) states geometrically and studied their properties for a simple choice of parameters. These simplifications does not hamper the qualitative geometrical interpretation of these states. Actually, consideration of all the parameters makes the results complicated and not easily visible in the phase space picture. Due to this reason we have simplified the results by these choices. We have also shown that two mode squeezing deteriorates the squeezing effect in angular momentum quadratures. We applied the SAMS states in analyzing the sensitivity of interferometry [14]. The effect of two mode squeezing on interferometry can be seen from the results of the uncertainties in two mode squeezing. As two mode squeezing increases the uncertainty of the squeezed quadrature it will also increase the value of minimum detectable phase difference ($\Delta\Phi$) of any interferometer using beam splitters. The relation between them in the frame rotated by $\frac{\pi}{2}$ about X axis can be written as $\Delta\Phi = \frac{\Delta J_y}{|\langle J_x \rangle|}$ [14].

Any ensemble of two quantum-level system (*e.g.* atoms or molecules can be considered as a spin-$\frac{1}{2}$ particle is described by the SU(2) algebra of angular momentum systems [6,7]. The number operators ($\hat{n}_{\pm} = j \pm m$) in the case of interferometric representation correspond to the population or occupation numbers in the upper and lower states of the system. In fact the complete set to describe the system is achieved by adding a permutation group P_N to the SO(3) group. However, this does not change the basic essence of the formalism. Squeezed atomic states were constructed by preferential population distributions [14]. Feynman *et al.* [4] have shown that the components of the pseudo angular-momentum vector completely specifies the state of the system semiclassically. The power of the geometrical method developed by Feynman *et al.*, lies in visualising and solving problems involving transition between two quantum levels. For example, the two classic problems discussed by them, the beam type maser oscillators and the radiation damping, could be visualised very clearly by the orientation of the state vector. Later application has led to elegant method for visualising and solving the photon echo problem also. Geometrical methods are found useful for problems that can be solved analytically. But it can also provide valuable insight into the behavior of the prowesses that are insolvable by analytical technique. Fermion interferometry can be considered by changing the bosonic commutation relation to the anticommutation relation for fermions. This is a totally new possibility as no fermion analog of the boson squeezed states has yet been found [17].

Arecchi [7] had developed the coherent atomic states by rotating the minimum uncertainty Dicke state (lowest projection state $|j, -j\rangle$) in three dimensional phase space. Recently, the authers in Ref. [9] have described the Dicke states as a cap and annular surfaces. As these states are eigenstates of J_z they should not be described by cap or annular surfaces with nonzero ΔJ_z on the mean sphere of the angular momentum vector. Instead the different Dicke states ($|j, m\rangle$) should be represented by an uncertainty circle. The Arecchi type atomic coherent states have some other geometrical representation problem wcich is not present in our picture. Their coherent states are actually some rotated angular momentum ground states. The projections of these rotated states in X-Y plane is an ellipse and thus show squeezing. Physically, mere rotation should not change the status of the state. This question has been raised by Kitagawa *et.al* [10] that if these Arecchi type coherent states describe squeezed angular momentum states under suitable choice of coordinates. Moreover, the area of the projected ellipse, which is a measure of the uncertainty product, is reduced from the area of the uncertainty circle of the ground state. This is a direct violation of Heisenberg uncertainty principle. In our definition of angular momentum coherent or squeezed states we have overcome this difficulty in representation and answered the question of Kitagawa *et al.*. The geometrical picture developed by us for SAMC or SAMS states does not have this ambiguity and thus is a better representation for the angular momentum coherent and squeezed states.

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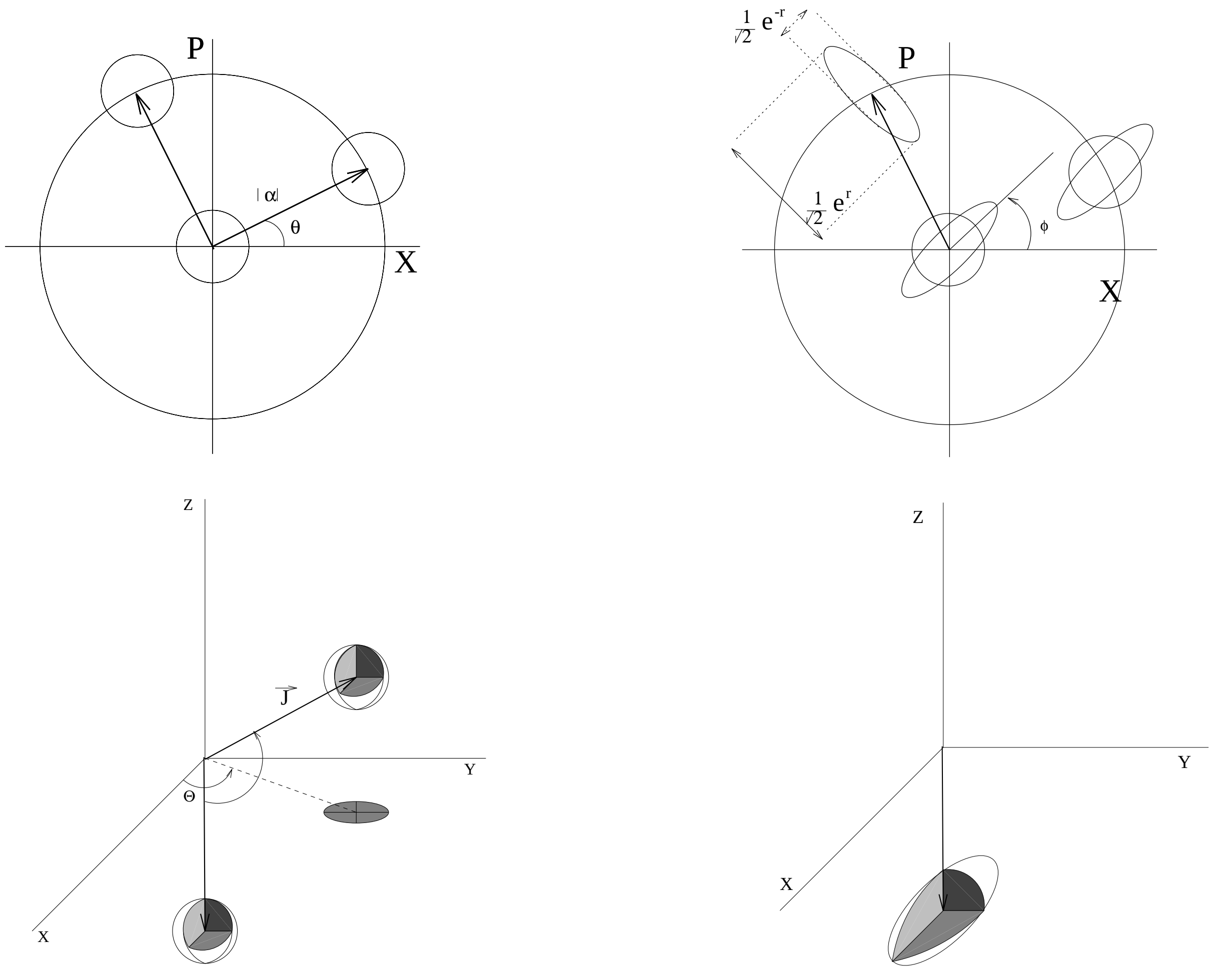

FIG. 1. Schematic phase space structures of uncertainties in angular momentum coherent and squeezed states are compared with the harmonic oscillator coherent and squeezed states referring to the uncertainties. The figures are not drawn according to scale. The figures for the harmonic oscillator are in two dimensional phase space whereas the figures for the angular momentum are in three dimensional phase space. The figures presented are : (a) harmonic oscillator vacuum or ground state (minimum uncertainty) with the uncertainty circle and coherent state obtained by displacing the ground state by a distance $|\alpha|$ towards the direction making an angle θ with the space coordinate; (b) harmonic oscillator squeezed state of vacuum or coheremt state with uncertainty ellipse having axes $\frac{e^{\pm r}}{\sqrt{2}}$; (c) angular momentum uncertainty sphere at two positions (one at pole and other in a rotated direction); (d) uncertainty ellipsoid for angular momentum squeezed state .

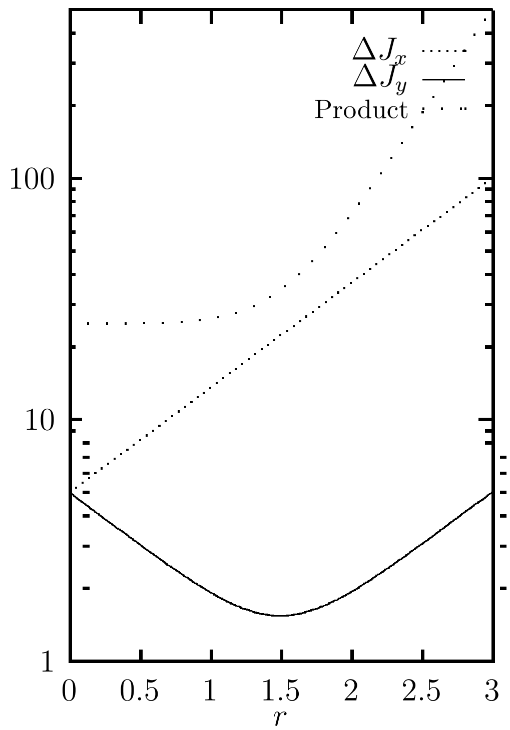

FIG. 2. Variation of J_x, J_y and their product with r for single mode squeezing with $j = 50, \delta$=0 and $m = -50$ (Some of the marking lines, at the bottom of the figure, are shifted towards right, which will be identified and corrected with the revised version(s).)

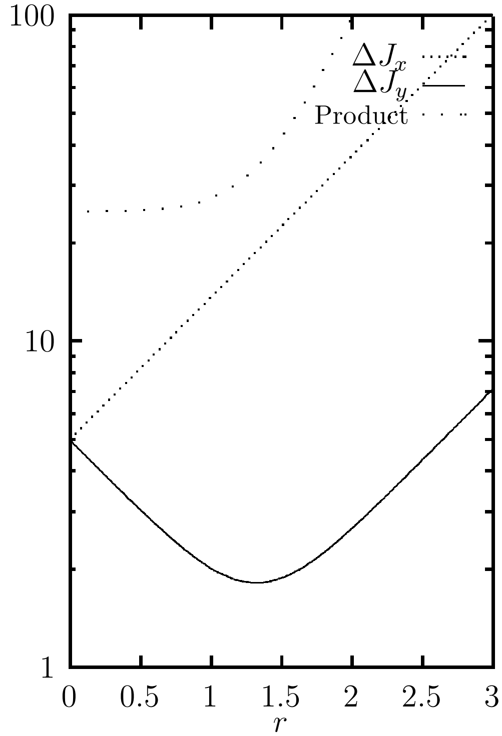

FIG. 3. Variation of J_x, J_y and their product with r for double mode squeezing with $j = 50, m = -50, \delta$=0 and $r_{\pm} = r$.